CBMS Issues in Mathematics Education
Volume **8**, 2000. pp. 128–153.

Why Can't Calculus Students Access their Knowledge to Solve Nonroutine Problems?

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ABSTRACT. In two previous studies we investigated the abilities of students just finishing their first year of a traditionally taught calculus sequence to solve nonroutine differential calculus problems. This paper reports on a similar study, using the same nonroutine calculus problems, with students who had completed one and one-half years of traditional calculus and were in the midst of an ordinary differential equations course. More than half of these students were unable to solve even one problem and more than a third made no substantial progress toward any solution. A test of associated algebra and calculus skills indicated that many of the students were familiar with the key calculus concepts for solving the nonroutine problems; nonetheless, students often used sophisticated algebraic methods rather than calculus in approaching the nonroutine problems. We suggest a possible explanation. These students may have had too few *tentative solution starts* in their *problem situation images* to help prime recall of the associated factual knowledge. We also discuss the importance of this for teaching.

1. Introduction

1.1. Background. Two previous studies demonstrated that students with C's as well as those with A's or B's in a traditional first calculus course had very limited success in solving nonroutine problems (Selden, Mason, and Selden, 1989; Selden, Selden, and Mason, 1994). Further, the second study showed that many of these students were unable to solve nonroutine problems for which they appeared to have an adequate knowledge base. This raised the question of whether more experienced students, those towards the end of a traditional calculus/differential equations sequence, would have more success; in particular, would they be better able to access and use their knowledge in solving nonroutine problems? Folklore has it that one only really learns material from a mathematics course in subsequent courses. The results reported here in part support and in part controvert this notion. As will be discussed, the differential equations students in this study often used algebraic methods (first introduced to them several years before their participation in the study) in preference to those of calculus courses taken more recently. These

We would like to acknowledge partial support from Chapman University, the ExxonMobil Education Foundation, Tennessee Technological University, and the National Science Foundation (under Grant #DGE-9906517).

students, who had more experience with calculus than those in the first two studies, appealed to sophisticated arithmetic and algebraic arguments more frequently than students in the earlier studies. Although somewhat more accomplished in their problem-solving ability, slightly more than half of them still failed to solve a single nonroutine problem, despite many having an apparently adequate knowledge base.

As in the previous two studies, what we are calling a nonroutine or novel problem is simply called a problem, as opposed to an exercise, in problem-solving studies (Schoenfeld, 1985). A *problem* can be seen as comprised of two parts: a task and a solver. The solver comes equipped with information and skills and is confronted with a cognitively non-trivial task; that is, the solver does not already know a method of solution. Seen from this perspective, a problem cannot be solved twice by the same person, nor is a problem independent of the solver's background. In traditional calculus courses most tasks fall more readily into the category of exercise than problem. However, experienced teachers can often predict that particular tasks will be problems for most students in a particular course, and tasks that appear to differ only slightly from traditional textbook exercises can become problems in this sense.

1.2. Overview of the Paper and Related Literature. In Section 2 we describe the setting and subjects – differential equations students who had taken a traditional calculus course obtaining grades of A, B, or C, at least one fourth of whom went on to obtain master's degrees and one a Ph.D. We present the two tests – one with five moderately nonroutine differential calculus problems, administered first, and a subsequent ten-question routine test of corresponding algebra and calculus skills. Section 3 contains a comparison of these students' performance on the two tests and introduces the notions of *full, substantial,* and *insubstantial factual knowledge.* In Section 4 we provide detailed information on the students' favored solution methods and compare these with what was observed in our previous two studies (Selden et al., 1989, 1994). Although the differential equations students were slightly more inclined to use calculus than students in the previous studies, they did so on only 39% of their solution attempts, preferring a combination of guessing, trial-and-error, arithmetic techniques, and algebra.

In Section 5 we analyze our results and suggest that students only slowly come to use their factual knowledge of calculus, or other mathematics, flexibly. Somewhat similar observations have also been made by Carlson (1998), Stacey and MacGregor (1997), and Dorier, Pian, Robert, and Rogalski (1998). While our differential equations students were quite ready to employ algebraic techniques, they were much less inclined, or able, to use calculus effectively. We introduce a nonroutineness scale for problems. Although nonroutineness is only one aspect of problems, the ability to solve moderately nonroutine problems is often seen as a hallmark of deep understanding of a the material in a course. We ask why our students could not solve our nonroutine problems and conjecture that they lacked a kind of knowledge, which when brought to mind produces what Mason and Spence (1999) have referred to as "knowing-to act in the moment." In describing this additional kind of knowledge we build on Tall and Vinner's (1981) idea of concept image. We introduce the notion of *problem situation image,* a mental structure associated with problem situations which may contain, amongst other things, *tentative solution starts,* i.e., various ways of beginning to solve a problem.

Finally some teaching implications of this conjecture are discussed in Section 6. We restrict our attention to moderately nonroutine problem-solving, referring readers to the work of Schoenfeld and others (Arcavi, Kessel, Meira, and Smith, 1998; Santos-Trigo, 1998; Schoenfeld, 1985) for a general treatment of problem-solving. We suggest that the construction of a problem situation, and its image, depends on student activities (experiences) analogous to the construction of a concept, as described by Breidenbach, Dubinsky, Hawks, and Nichols (1992) and Sfard (1991).

The central question we ask is: Why could our students not access their knowledge of calculus when needed? This question, of how one comes to know to act in a given situation, has been largely neglected in the mathematics education research literature. We offer a conjecture: what is missing is a kind of knowledge – tentative solution starts, ways of beginning, that are part of an individual's problem situation image. Such knowledge arises from a habit of mind, that of reflecting on various possible starting points. Yet, how does such knowledge come to mind in the moment? This is a question of how one brings information from long-term memory (one's knowledge base) into short-term memory (see Baddeley, 1995) and makes it conscious (see, for example, James, 1910 or Mangan, 1993). We suggest that recognizing a problem situation partly activates the information in its image which then primes the recall of factual knowledge.

2. The Course, the Students, and the Tests

2.1. The Calculus/Differential Equations Sequence. The setting is a southeastern comprehensive state university having an engineering emphasis and enrolling about 7500 students – the same university of the earlier studies of C and A/B first-term calculus students (Selden et al., 1989, 1994). The annual average ACT composite score of entering freshman is slightly above the national average for high school graduates, e.g., in the year the data were collected the university average was 21.1, compared to the national average of 20.6.

A large majority of students who take the calculus/differential equations sequence at this university are engineering majors. The rest are usually science or mathematics majors. A separate, less rigorous, calculus course is offered for students majoring in other disciplines.

Until Fall Semester 1989, the calculus/differential equations sequence was offered as a five-quarter sequence of five-hour courses. Since then, it has been offered as a four-semester sequence. Under both the quarter and the semester systems it has been taught, with very few exceptions, by traditional methods with limited, if any, use of technology and with standard texts (Swokowski (1983), Berkey (1988), or Stewart (1987), for calculus and Zill (1986) for differential equations. Class size was usually limited to 35–40 students, but some sections were considerably smaller and a few considerably larger. All but three sections were taught by regular, full-time faculty of all ranks; the three exceptions were taught by a part-time associate professor who held a Ph.D. in mathematics. All instructors taught according to their usual methods and handled their own examinations and grades.

2.2. The Students. The pool of 128 differential equations students considered in this study came from all five sections of differential equations taught in

Spring 1991, omitting only a few students who had taken an experimental calculator-enhanced calculus course or who were participants in the two previous studies. All students in this pool had a grade of at least C in first-term calculus.

In the middle of the Spring semester, all of the 128 beginning differential equations students were contacted by mail and invited to participate in the study. As with the previous study of A/B calculus students (Selden et al., 1994), each student was offered $15 for taking the two tests and told he or she need not, in fact should not, study for them. The students were told that three groups of ten students would be randomly selected according to their first calculus grades and in each group there would be four prizes of $20, $15, $10, and $5. The latter was an incentive to ensure that all students would be motivated to do their best. Altogether 11 A, 14 B, and 12 C students volunteered and ten were randomly selected from each group. Of those, 28 students (nine A, ten B, and nine C) actually took the tests: three mathematics majors and 25 engineering majors (nine mechanical, five chemical, four civil, four electrical, two industrial, and one undeclared engineering concentration). These majors reflected the usual clientele for the calculus/differential equations sequence.

TABLE 1. Fall 1990 Calculus III grades for study participants and for all students enrolled in the course.

Grade	Participants		All Students	
A	4	(17%)	8	(5%)
B	6	(27%)	24	(15%)
C	7	(30%)	49	(30%)
D	4	(17%)	41	(25%)
F	2	(9%)	33	(20%)
W	0	(0%)	10	(6%)
Total	23	(100%)	165	(100%)

At the time of the study, all but one of the 28 students tested had taken the third semester of calculus at this university; the one exception was enrolled in Calculus III and Differential Equations simultaneously. Their grades in Calculus III were 5 A, 8 B, 8 C, 4 D, and 2 F. Of these, one D student and one F student were repeating Calculus III while taking Differential Equations. Twenty-three of the students had taken Calculus III in the immediately preceding semester (Fall 1990). Their grades and the grades of all students who took Calculus III that semester are given in Table 1, which indicates that the better mathematics students are over-represented in this study.

In Table 2 we give the mean ACT scores and the mean cumulative grade point averages (GPA) at the time of the test for the 28 students in this study and compare this with the same information for the students in our two earlier studies (Selden et al., 1989, 1994). The numbers for the Differential Equations (DE) students are quite close to those of the A/B calculus students but considerably above those of the C calculus students.

Eleven of the 28 differential equations students in this study had already taken additional mathematics courses. Of the three mathematics majors, two had completed, and the third was then currently taking, a "bridge to proof" course, and the

TABLE 2. Mean ACT and GPA of students in all three studies.

Study	Mean ACT	Mean Math ACT	Cumulative GPA
DE	26.26	27.74	3.145
(A/B) Calculus	27.12	28.00	3.264
(C) Calculus	24.18	25.65	2.539

third had also taken Discrete Structures. In addition, five students had taken Complex Variables, and another was enrolled in that course at the time of the study. One of these five had also taken an introductory matrix algebra course, as had two other students. Except for one C, all grades for these students in these additional mathematics courses were at least B. In our analysis of the results we will compare the students who had studied mathematics beyond the calculus/differential equations sequence with those who had not.

Of the 28 students in this study, 22 (79%) graduated within six years of their admission to the university as first-year students. In comparison, for the university as a whole over the same time period, the average graduation rate within six years of admission was 41%.

As of May 1999, it was known that all but two of the 28 students tested had earned bachelor's degrees at this university, three in mathematics and the others in engineering. In addition, five students had earned master's degrees in engineering and one had earned an MBA, all at this university. One student had earned a master's degree in mathematics at this university and a Ph.D. in mathematics at another university. There may be additional accomplishments of these kinds among the 28 students, but they could not all be traced.

The students in this study represented 33% of the A's, 26% of the B's, and 6% of the C's in Differential Equations that semester, and none of the 30 D's, F's and W's. In addition, after a minimum of three semesters at this university, these 28 students had a mean GPA of 3.145 for all courses taken. Their graduation rate was almost double that of the university as a whole, and at least 25% of them went on to complete a graduate degree. By all these indicators, at the time of the study and subsequently, these students were among the most successful at the university.

2.3. The Tests. The items for the nonroutine test were originally chosen for the study of average calculus students' problem solving (Selden et al., 1989). Problems were chosen which could be solved using material covered in the first term of differential calculus. That the five nonroutine problems were, in fact, novel for those students was determined by inviting department faculty to an informal seminar where possible problems were presented and the faculty were asked for suggestions. To the best of our knowledge, these problem types had not been taught or assigned in any of the classes that year. Solutions to the problems are no more complex than those traditionally covered in the university's calculus courses. However, in order to make progress towards a solution, students must access and combine ideas in ways that are new.

Students were allowed one hour to take the five-problem nonroutine test, followed by half an hour for a ten-part routine test comprised of associated algebra and calculus exercises. Prior to the nonroutine test, the students were told they might find some of the problems a bit unusual. No calculators were allowed. They were asked to write down as many of their ideas as possible because this would be helpful to us and to their advantage. They were told that A students (in first calculus) would only be competing against other A students for prizes, and similarly, for B and C students. They were assured all prizes would be awarded and partial credit would be given.

Each nonroutine problem was printed on its own page, on which student work was to be done. All students appeared to work diligently for the entire hour.

As soon as the nonroutine tests were collected, the students were given the two-page routine test. They were told answers without explanations would be acceptable, but they could show their work if they wished. Most students worked quickly, taking from 12 to 17.5 minutes to complete the routine exercises. None stayed the allotted half hour.

2.3.1. **The Nonroutine Test.**

1. Find values of a and b so that the line $2x + 3y = a$ is tangent to the graph of $f(x) = bx^2$ at the point where $x = 3$.
2. Does $x^{21} + x^{19} - x^{-1} + 2 = 0$ have any roots between -1 and 0? Why or why not?
3. Let $f(x) = \begin{cases} ax, & x \leq 1 \\ bx^2 + x + 1, & x > 1 \end{cases}$. Find a and b so that f is differentiable at 1.
4. Find at least one solution to the equation $4x^3 - x^4 = 30$ or explain why no such solution exists.
5. Is there an a such that $\lim_{x\to 3} \frac{2x^2 - 2ax + x - a - 1}{x^2 - 2x - 3}$ exists? Explain your answer.

2.3.2. **The Routine Test.**

1. (a) What is the slope of the line tangent to $y = x^2$ at $x = 1$?
   (b) At what point does the tangent line touch the graph of $y = x^2$?
2. Find the slope of the line $x + 3y = 5$.
3. If $f(x) = x^5 + x$, where is f increasing?
4. If $f(x) = x^{-1}$, find $f'(x)$.
5. (a) Suppose f is a differentiable function. Does f have to be continuous?
   (b) Is $f(x) = \begin{cases} x, & x > 0 \\ 2, & x \leq 0 \end{cases}$ continuous?
6. Find the maximum value of $f(x) = -2 + 2x - x^2$.
7. Find $\lim_{x\to 1} \frac{x^2 - 1}{x - 1}$.
8. Do the indicated division: $x - 1 \overline{\smash{)}\, x^3 - x^2 + x - 1}$.
9. If 5 is a root of $f(x) = 0$, at what point (if any) does the graph of $y = f(x)$ cross the x-axis?
10. Consider $f(x) = \begin{cases} x^2, & x \leq 1 \\ x + 3, & x > 1 \end{cases}$.
    (a) Find $\lim_{x\to 1^+} f(x)$.
    (b) What is the derivative of $f(x)$ from the left at $x = 1$ (sometimes called the left-hand derivative)?

As in the previous studies (Selden et al., 1989, 1994), each nonroutine test problem was assigned 20 points and graded by one of the authors and checked by the others. If the student's work showed substantial progress toward a correct solution then at least 10 points were awarded. Arithmetic errors reduced scores by 1 point. The mean score on the nonroutine test was 21.3, as compared with 20.4 for the A/B and 10.2 for the C calculus students. The lowest nonroutine score in all three studies was zero.

On the routine test each question was assigned 10 points. Again, one point might be lost for an arithmetic or representational error.[1] The highest routine test score was 100 (out of 100) and the lowest score was 50, as compared to a high score of 90 and a low score of 53 for the A/B calculus students (Selden et al., 1994). The mean score on the routine test was 75.3.

Each problem on the nonroutine test could be solved using a combination of basic calculus and algebra skills. The correspondence between routine questions and nonroutine problems is given in Table 3.

TABLE 3. Correspondence between routine questions and nonroutine problems.

Nonroutine problem	1	2	3	4	5
Corresponding routine questions	1, 2	3, 4, 9	5, 10	6, 9	7, 8

3. The Results

We present the test results from several different perspectives. We examine the students' ability to solve nonroutine problems, their knowledge base (of associated basic calculus and algebra skills), and whether they were able to access and use their resources effectively.

3.1. Ability to Solve Nonroutine Problems. Slightly more than half (57%) the differential equations students (16 of 28) failed to solve a single nonroutine problem. This is somewhat better than the two-thirds of A/B, and all of C, first-year calculus students who failed to solve a single nonroutine problem in the previous studies (Selden et al., 1989, 1994).

In order to analyze the nonroutine test results, we make a distinction between a solution *attempt,* a page containing written work submitted as a solution to a nonroutine problem, and a solution *try,* any one of several distinct approaches to solving the problem contained within a single solution attempt. In only four instances did a student not attempt a nonroutine problem; thus there were 136 attempts by the 28 students on the five nonroutine problems. On these attempts there were a total of 243 solution tries. Of the 136 attempts, 20 were judged *completely correct* (except possibly for a minor computational error). Twelve other solution attempts were found to be *substantially correct* because they exhibited substantial progress toward a solution, that is, the proposed solution could have been altered or completed to arrive at a correct solution.

Of the 20 completely correct solutions, five were for Problem 1, three for Problem 2, five for Problem 3, none for Problem 4, and seven for Problem 5. These completely correct attempts came from 12 of the 28 students; the 12 substantially

[1] An answer of 5 on Routine Problem 9 received full credit as most students did not seem to distinguish between "meet" and "cross."

correct attempts came from 11 students. Altogether, 18 of the 28 students provided at least one substantially or completely correct solution attempt. That is, 36% of the differential equations students (10 of 28) were unable to make substantial progress on any nonroutine problem; this is lower than the 42% and 71% reported previously for A/B and C first-year calculus students (Selden et al., 1989, 1994). Thus, the differential equations students did perform somewhat better than the first-year calculus students.

3.2. Comparison of Nonroutine and Routine Test Results: Did Students Have Adequate Resources and Use Them? The routine test was designed to determine whether the students' inability to do the nonroutine problems was related to an inadequate knowledge base of calculus and algebra skills (i.e., Schoenfeld's (1985, 1992) "resources"). Did the students lack the necessary factual knowledge or did they have it without being able to access it effectively? Scores on the corresponding routine questions (shown in Table 3) were taken as indicating the extent of a student's factual knowledge regarding a particular nonroutine problem. As in our 1994 study, a student was considered to have *substantial factual knowledge* for solving a nonroutine problem if that student scored at least 66% on the corresponding routine questions. A student was considered to have *full factual knowledge* for solving a nonroutine problem if that student's answers to the corresponding routine questions were correct, except possibly for notation, for example, answering $(1, -1)$ instead of -1 to Question 6. All others were considered as having *insubstantial factual knowledge.*

Table 4 gives the number of completely or substantially correct solutions for nonroutine problems by solver's factual knowledge (as demonstrated on the corresponding routine questions). For example, on Problem 1, 15 (of 28) students had full factual knowledge; of these, five gave completely correct and two gave substantially correct solutions. An additional ten (of 28) students had substantial factual knowledge for Problem 1, but none of them gave completely or substantially correct solutions. That is, the performance of these ten students on the routine questions seemed to indicate they had sufficient factual knowledge to solve, or at least make substantial progress on, Problem 1; yet they either did not access it or were unable to use their knowledge effectively to make progress. The remaining three students demonstrated insubstantial factual knowledge. Thus, a total of seven students gave completely or substantially correct solutions on Problem 1.

Taking another perspective, in the 59 routine test solution attempts in which students demonstrated full factual knowledge, they were able to solve the corresponding nonroutine problem 14 times (24%) or make substantial progress towards its solution six times (10%). Thus, on slightly more than a third of their attempts (34%), these students accessed their knowledge effectively. Students with substantial, but not full factual knowledge, did so on less than a quarter of their attempts. These results are summarized in Table 5. Furthermore, six students showed *no* factual knowledge of the components necessary for a nonroutine problem and they each had a score of zero on the corresponding nonroutine problem.

In order to compare overall student performances on the routine and nonroutine tests, we introduce the notion of a *score pair,* denoted $\{a, b\}$, where a is the student's score on the routine test and b is the student's score on the nonroutine test. In every case, $a > b$. Figure 1 shows students' routine test scores in ascending

TABLE 4. Number of correct solutions for nonroutine problems by solver's factual knowledge.

	Nonroutine problem				
	1	2	3	4	5
Full factual knowledge	*15*	*3*	*5*	*14*	*22*
Problem completely correct	5	1	1	0	7
Problem substantially correct	2	0	1	0	3
Substantial factual knowledge	*10*	*19*	*12*	*1*	*2*
Problem completely correct	0	2	4	0	0
Problem substantially correct	0	3	1	0	0
Insubstantial factual knowledge	*3*	*6*	*11*	*13*	*4*
Problem completely correct	0	0	0	0	0
Problem substantially correct	0	0	1	0	1
Total completely or substantially correct	7	6	8	0	11

TABLE 5. Percentage of correct solutions to nonroutine problems from students with the requisite factual knowledge.

	Full factual knowledge (59)	Substantial factual knowledge only(44)
Completely correct nonroutine solution	24% (14/59)	14% (6/44)
Substantially correct nonroutine solution	10% (6/59)	9% (4/44)

order (from left to right); superimposed below each student's routine test score is her/his nonroutine test score.

The three students with the highest nonroutine scores had score pairs of $\{90, 69\}$, $\{85, 59\}$, and $\{83, 59\}$; the first of these subsequently obtained a B.S. in civil engineering, summa cum laude with a cumulative grade point average of 3.948. The three mathematics majors in the study, all of whom had completed at least one additional mathematics course at the time of the study, had score pairs $\{95, 34\}$, $\{89, 18\}$ and $\{87, 21\}$. That is, they scored in the top quarter on the routine test, and taken together, scored slightly higher than the mean nonroutine score of 21.3. The last of these three subsequently obtained a Ph.D. in mathematics from a major state university.

Student performance on the routine and nonroutine tests was not improved by having studied additional mathematics. The respective mean scores for the eleven students who had done so were 73.1 (vs. 75.3 for all of the students) and 15.3 (vs. 21.3 for all of the students).

Figure 1 shows a positive correlation (coefficient $r = 0.68$) between factual knowledge (resources) and the ability to solve novel problems. Nonetheless, having the resources for a particular problem is not enough to assure that one will be able to solve it. Two students had score pairs of $\{86, 4\}$ and $\{80, 3\}$, suggesting that having a reasonably good knowledge base of calculus and algebra skills (resources) is not sufficient to make substantial progress on nonroutine calculus problems. One student, score pair $\{83, 59\}$, lacked substantial factual knowledge on only those routine questions associated with Problems 2 and 4 (on which he scored zero) and

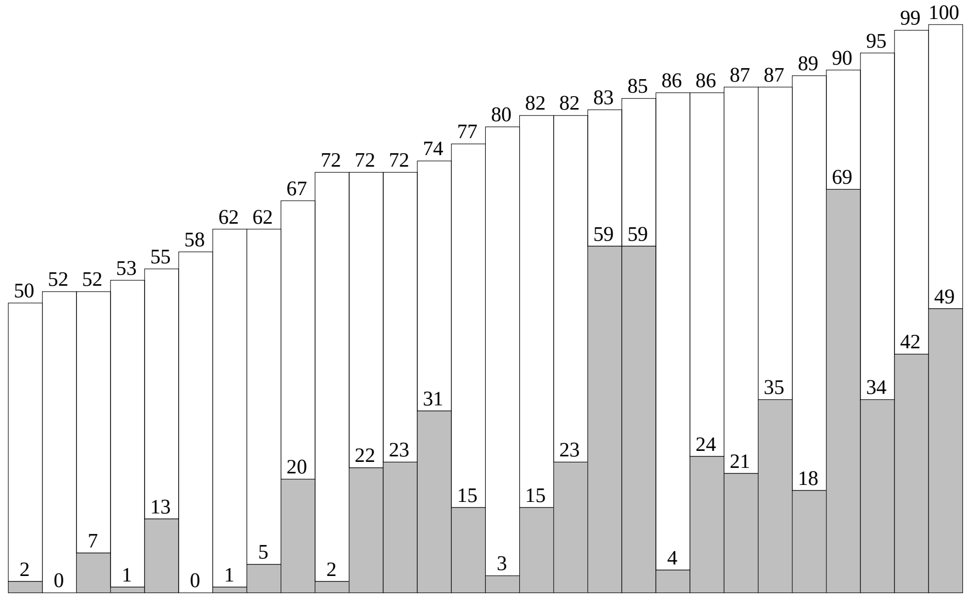

FIGURE 1. Upper score is for the routine test (factual knowledge). Lower score is for the nonroutine test.

solved the three remaining nonroutine problems (1, 3 and 5) completely correctly. This student and one other, the $\{90, 69\}$ score pair, were the only students whose performance on the routine questions appeared to correspond closely with the nonroutine problems they solved. An analysis of solution attempts indicates that some students were hampered by misconceptions. Indeed, the literature (Amit and Vinner, 1990; Eylon and Lynn, 1988) suggests that misconceptions are more likely to surface during attempts to solve nonroutine problems, a phenomenon observed in this study and discussed in Section 4.

4. Favored Solution Methods

4.1. Nonroutine Problem 1. *Find values of a and b so that the line $2x + 3y = a$ is tangent to the graph of $f(x) = bx^2$ at the point where $x = 3$.*

Fifteen students rewrote the equation of the line in the form $y = \frac{a}{3} - \frac{2}{3}x$, set the right-hand side of this equation equal to that of the parabola and solved for either a or b. Eight of these ignored the tangency of the line to the curve. The remaining seven also set the derivative of f equal to the slope of the line to obtain a second equation so a and b could be fully determined. These were the seven with completely or substantially correct solutions. An additional seven students took a derivative, but abandoned it at some point.

The two most frequently occurring misconceptions on Problem 1 involved the meaning of tangent line. Six students conflated the ideas of the equation of the tangent line, the slope of the tangent line, and the derivative of the function. Two other students incorrectly claimed that the tangent line was perpendicular to the curve at the point of tangency and used the negative reciprocal of the derivative for the slope of the tangent line. Missing among these students was the error found among A/B calculus students of confusing a secant line, calculated using two points on the parabola, with the tangent line.

4.2. Nonroutine Problem 2.

Does $x^{21} + x^{19} - x^{-1} + 2 = 0$ *have any roots between* -1 *and* 0*? Why or why not?*

On Problem 2, the three correct, and three substantially correct, solutions made no use of calculus. Instead, all used sophisticated arithmetic and algebraic reasoning to compare the relative sizes of the component terms in the given polynomial. The correct solutions made use of the observation that, for values of x in the open interval $(-1,\ 0)$, both $-x^{-1}$ and $x^{21} + x^{19} + 2$ would be positive and hence their sum must be positive. This type of first-principles argument was also used, although less successfully, by about the same proportion of the A/B first-calculus students.

Other solution attempts on Problem 2 suggest that these differential equations students were relatively comfortable with, and knowledgeable about, algebraic techniques. Yet their knowledge included some common misconceptions. Two students inappropriately applied Descartes' Rule of Signs, and eight erroneously concluded that no roots could exist when the Rational Root Test produced no solutions. Even this flawed use of algebra is an improvement upon the favored solution method of the C calculus students: substitute values for x until becoming convinced that no guess is ever going to work, hence no root exists (a method also used by four of the differential equations students). Only five of the differential equations students used any calculus on Problem 2, four taking derivatives but not using them. The fifth student did make effective use of the derivative but made an unrelated error.

4.3. Nonroutine Problem 3.

Let $f(x) = \begin{cases} ax, & x \leq 1 \\ bx^2 + x + 1, & x > 1 \end{cases}$. *Find* a *and* b *so that* f *is differentiable at* 1.

Ten students set $ax = bx^2 + x + 1$, eight of them substituting $x = 1$ to get the relationship $a = b + 2$, and then stopped. Two of these ten students expressed doubts about the completeness of their solutions. One student had seven solution tries, all variations on the theme of matching for continuity. Eleven of the 26 who attempted a solution to Problem 3 took a derivative. Several differentiated ax and got x. Of the eleven who used calculus, eight made at least substantial progress towards a solution; five used the derivative to arrive at a completely correct solution and three used it to obtain a substantially correct solution.

4.4. Nonroutine Problem 4.

Find at least one solution to the equation $4x^3 - x^4 = 30$ *or explain why no such solution exists.*

There were no completely correct or substantially correct solutions. Since these traditionally-taught calculus students were not allowed to use graphing, or other, calculators in this study, no easy graphical methods were available to them. Of the 27 students who attempted this problem, 44% (12 of 27) used the same method: narrow the domain of possible x values by eliminating those for which $4x^3 - x^4$ cannot be close to 30. Most of these students determined that no solutions could exist outside of the open interval $(1,\ 4)$ and some also eliminated the integer values of $1, 2$, and 3. This 44% who used sophisticated algebraic and arithmetic reasoning exceeds the 26% (5 of 19) of A/B calculus students who did so.

The next most popular solution method, used by six students (22%), was the Rational Roots Test. However, all six students incorrectly concluded that if a rational root could not be found from the factors of 30 then no roots at all existed. In our earlier study of A/B calculus students, this approach was also used by 20% of the students. Only two students used the method favored by C calculus students in the earliest study: factor $4x^3 - x^4$ and set each factor equal to 30. Four students took the first derivative, set it equal to zero and stopped. Several students used synthetic division to check whether particular values were roots.

4.5. Nonroutine Problem 5.

Is there an a such that $\lim_{x\to 3} \dfrac{2x^2 - 2ax + x - a - 1}{x^2 - 2x - 3}$ *exists? Explain your answer.*

On Problem 5, 39% of the solution attempts (11 of 28) were substantially or completely correct, similar to the 37% (7 of 19) of A/B calculus student attempts with substantial progress towards a solution.

Of the 28 solution attempts, 15 involved the use of L'Hôpital's Rule. Nine of these 15 attempts were at least partially successful. In the other six instances students failed to note that the numerator as well as the denominator must have limit zero before applying L'Hôpital's Rule. Five students substituted 3 for x, found the denominator of the expression to be zero and asserted that no limit could exist since the denominator was zero at the limiting value of the variable (two of these were mathematics majors). This was the favored method of the C calculus students (47% of them used it) and was used by 26% of the A/B calculus students. Here it was found in just 18% (5 of 28) of the solution attempts. Six students struggled, algebraically, with finding a way to factor the numerator so that the $(x - 3)$ in the factored denominator could be canceled and the limit taken; two of these resulted in completely correct solutions.

4.6. Summary.

4.6.1. *Use of calculus.* Inasmuch as Nonroutine Problem 5 involved evaluating a limit, any solution attempt could be considered to involve calculus, we omit it from the following analysis. On Nonroutine Problems 1, 2, 3, and 4, the differential equations students used calculus – often taking a derivative – on fewer than half (39%) of all solution attempts (42 of 108). Furthermore, in fewer than half of these (16 of the 42), students used calculus to produce potentially useful information, i.e., it could have been used to make progress towards a correct solution. Fifteen of the 16 potentially useful solution attempts led to substantially or completely correct solutions. That is, about three-fourths (15 of 21) of the completely or substantially correct solutions to Nonroutine Problems 1, 2, 3, and 4 made effective use of calculus.

We observe that the differential equations students were no more inclined to resort to calculus than were the A/B and C calculus students of the previous two studies. They did not use calculus on 61% of these attempts (66 of 108); this is essentially the same percentage as in the previous studies (61% for the A/B and 59% for the C calculus students).

However, it appears that as students proceed through calculus to differential equations, the number of calculus misconceptions decreases. Slowly they become more proficient (or perhaps the less competent students drop out). For example, incorrectly asserting on Problem 5 that the limit of a quotient cannot exist when the denominator is 0 went from 47% for C calculus students, to 26% for the A/B calculus students, to just 18% for the differential equations students.

4.6.2. *More sophisticated, but still flawed, use of algebra.* Considering now all five nonroutine problems, on 56% of the solution attempts calculus was not used; rather, a combination of guessing, trial-and-error, arithmetic techniques, and algebra was used. Eleven of the 32 substantially or completely correct solutions were almost purely algebraic, seven on Problem 2, one on Problem 4, and three (after the student observed the need to reduce the fraction in order to take the limit) on Problem 5. While these solutions demonstrated a level of algebraic competence not found in the first-year calculus students of our earlier studies, quite a few other attempts to use algebra were flawed. For example, fourteen solution attempts included an improper use of the Rational Root Test and two solution attempts used Descartes' Rule of Signs in an inappropriate setting.

4.6.3. *Use of graphs.* Most graphing was done by students on Problem 1 – only four other graphs, three incorrect, appeared in all of the solution attempts for the other problems. Of the 27 students who attempted to solve Problem 1, 22 (81%) sketched at least one graph and six (22%) sketched three or more; this is substantially higher than the 12 of 19 (63%) who used graphs in the A/B study, where only 1 (5%) sketched more than two graphs. Fourteen of the 22 who used graphs in the present study first drew some version of the graph pictured in Figure 2. Of those 14 students, six also produced a correct graph like that in Figure 3. Three of these six rejected an incorrect graph (by striking through it); giving one substantially correct and two completely correct solutions to Problem 1, perhaps an example of monitoring their work (Schoenfeld, 1985, 1992). A fourth student drew only the correct graph, the one pictured in Figure 3, and gave a completely correct solution.

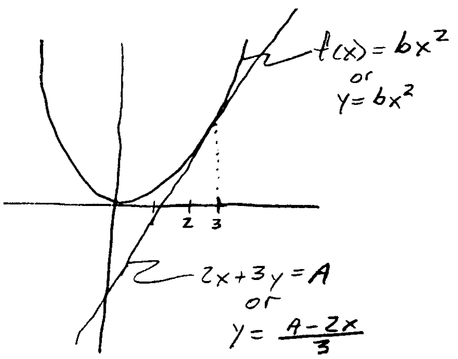

FIGURE 2. Example of frequently sketched incorrect graph for Problem 1.

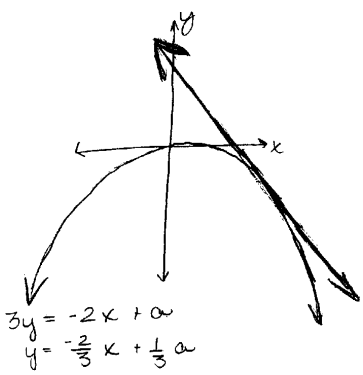

FIGURE 3. Example of a correctly sketched graph for Problem 1.

The fifth completely correct solution for Problem 1 came from a student who drew no graphs at all. Hence, in this study, four of the five completely correct solutions showed a graph while only one of the three completely correct solutions in the previous study of A/B students was accompanied by a graph. The other three differential equations students who drew both graphs used little calculus and did not indicate any rejection of incorrect graphs; in fact, all three seemed to be considering various cases for values of a and b by drawing a variety of graphs (between them, these three students produced ten graphs). The others who drew graphs were either misled by the assumption that the parabola was concave up (e.g. as in Figure 2) or gained no useful information from their graphs.

Whereas students who successfully solved Problem 1 in the present study were willing to draw and reject graphs, those in the A/B study who chose to use graphs generally had only one – either correct accompanied by a substantially or completely correct solution or incorrect and accompanied by an incorrect solution. In the A/B study, 16 graphs were produced by 12 students on Problem 1 whereas in the present study, 37 graphs were drawn by 22 students. Three of the 12 A/B students who used graphs (25%) rejected (crossed out) one of their graphs (including one who rejected a graph which was correct) while six of the 22 (27%) who used graphing on Problem 1 in this study rejected a graph. It would appear, then, that although the more experienced students were more willing to consider graphical ideas than the less experienced students (81% versus 63%), they were about equally likely to abandon the graphs they produced. This provides some evidence that extra experience in a traditional classroom environment does not necessarily increase effective self-regulation during problem solving (Boaler, 1999; Schoenfeld, 1992).

DeFranco's (1996) paper on expert problem solvers with Ph.D.'s in mathematics suggests that the skills possessed by experts which are often lacking in nonexperts might include a willingness to create, *abandon,* and *revisit* many ideas in the solution process. Thus it might be useful to know how students develop a willingness to risk committing graphical, and other ideas, to paper and to reject such ideas once they have been given life on paper.

5. Analysis

Reflecting on the three studies, one wonders when, if ever, traditionally taught students learn to use calculus flexibly enough to solve more than a few nonroutine problems. Furthermore, how could it happen that students, who were more successful than average by a variety of traditional measures and who demonstrated full factual knowledge for a nonroutine problem, failed to access and use their knowledge successfully on 76% of their attempts (as shown in Table 5)? Many of these students (the engineering majors) had almost completed their formal mathematical educations, except possibly for one or two upper-division mathematics courses, leaving them limited opportunity in future mathematics courses to improve their abilities to solve nonroutine problems. Finally, does it matter whether students can solve such problems?

5.1. When do students finally learn to apply calculus flexibly? It would appear from this sequence of three studies that, at least for traditionally-taught calculus students in classes of 35–40, the ability to solve nonroutine calculus problems develops only slowly. Performance for the best students went from one third who could solve at least one nonroutine beginning calculus problem toward the end of their first year of college calculus to slightly less than half who could do so toward the end of the two-year calculus/differential equations sequence. In addition, the percentage of correct solutions increased slightly over the three studies (see Table 6).

For the sake of comparison, consider for a moment only those students in this study who had a grade of A or B in first-term calculus. There were 19 of them, nine

TABLE 6. Percent of correct solutions in all three studies.

Study	Completely correct	Substantially correct
DE	14% (20/140)	9% (12/140)
(A/B) Calculus	9% (9/95)	9% (9/95)
(C) Calculus	0% (0/85)	7% (6/85)

A and ten B. In the previous study of A/B calculus students there were ten A and nine B students. In the two groups of students there was very little difference in the number who did not answer any nonroutine problem at least substantially correctly (8 in the previous study, 6 in this one). In Figure 4 we show the distribution of students according to how many nonroutine problems each answered. For this figure, a substantially correct solution was counted as 0.5 and a completely correct solution as 1; e.g., the one student in the A/B study who gave two completely correct and two substantially correct solutions is represented by the single box to the left of the 3 in Figure 4. In Figure 5 we give similar histograms for just the completely correct solutions. Only two of the completely correct solutions in this (the DE) study came from students who had a C in first calculus, the other 18 were from A and B students. In both figures, the incremental shift upward is apparent, but not significant. [2]

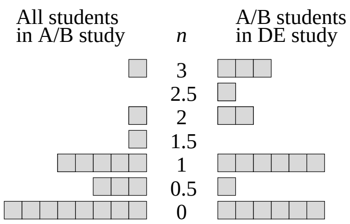

FIGURE 4. Number of students with at least n substantially correct solutions.

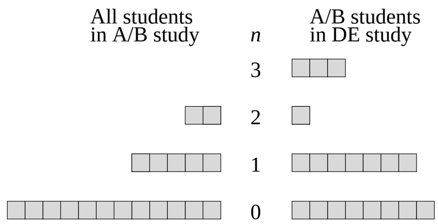

FIGURE 5. Number of students with n completely correct solutions.

Notably, by the time these students were coming to the end of their calculus/differential equations sequence their algebra skills seemed to be relatively sophisticated and readily accessible, albeit somewhat flawed. Such slow, incremental growth in mathematical capabilities may not be unusual. In a cross-sectional study of students' development of the function concept, Carlson (1998) investigated students who had just received A's in college algebra, second-semester calculus, or first-year graduate mathematics courses. She found that "even our best students do not completely understand concepts taught in a course, and when confronted with an unfamiliar problem, have difficulty accessing recently taught information. . . . Second-semester calculus students had a much more general view of functions

[2]The mean number of nonroutine problems at least substantially correct per student for the A/B study was $\bar{x}_1 = 0.68$ (with standard deviation $s_1 = 0.82$) whereas for this study the mean was $\bar{x}_2 = 1.16$, (with $s_2 = 1.11$). The result of a Wilcoxon test on the hypothesis $H_0 : \bar{x}_2 = \bar{x}_1$ is $p \simeq 0.2$. For the comparison of completely correct solutions, a Wilcoxon test on $H_0 : \bar{x}_2 = \bar{x}_1$ gives $p \simeq 0.16$. These p-values are too large to say there is a statistically significant improvement.

[than college algebra students, but] . . . they were unable to use information taught in early calculus."

In addition, it is not unusual for students to fall back on earlier mathematical techniques with which they are perhaps more familiar and more comfortable. In a study of 900 Australian high school students in Years 9 and 10 ("in their third or fourth year of algebra learning"), Stacey and MacGregor (1997) found that when asked to solve three simple word problems, a large proportion wrote no equations, even though specifically asked to, and others tried to write equations but then switched to non-algebraic methods including trial-and-error and arithmetic reasoning to solve the problems.

In an analogous fashion, the differential equations students in this study relied more often (in 76 of 136 solution attempts) on a variety of arithmetic and algebraic techniques than on calculus. Taken together, our three studies suggest the folklore that one only really learns a course's material in the next course appears to be not quite accurate, rather several additional courses may be necessary. The differential equations students seemed most comfortable with algebraic methods – ideas first introduced to them several years before. Perhaps during those intervening years they had been exposed to numerous algebraic (sub)problems and built up a familiarity with algebraic techniques and habits of mind.

As in our previous two studies, over half of the students made no use whatever of calculus. This gives a negative answer to the question we posed in 1994: Would students at the end of a traditional calculus/differential equations sequence be more inclined to use calculus techniques in solving problems? Most of the students in this study had not learned to apply beginning calculus flexibly by the end of the calculus-differential equations sequence and many might well never do so.

The idea that students might or might not apply calculus, or any mathematics *flexibly* has been discussed by Dorier et al. (1998). They introduced three levels of applying mathematical knowledge to tasks: a *technical level* in which students are asked to apply calculus skills and use definitions, properties and theorems directly; a *mobilizable level* in which students adapt their knowledge to tasks which are not direct applications, require several steps, require some transformation or recognition that a property or theorem is to be applied; an *available level* in which students solve problems without being given an indication of methods, or must change representations. Dorier and colleagues tested one hundred French university students who had graduated (*Licence*) and were preparing for the CAPES competitive examination for teaching at secondary level (which just one in eight pass and which has curriculum the same as that for mathematics majors in the first two years of university). Dorier et al. found that whenever problems were anything but the technical level, the success rate was under 10%. These findings seem comparable to those in this study regarding students, who at the end of their calculus-differential equations sequence, produced just 14% completely correct solutions to our nonroutine problems (Table 6).

5.2. How does it happen that students can have the knowledge, but not be able to access and effectively use it to solve nonroutine problems? Part of the rationale for having our students take the routine test after the nonroutine test was to determine whether they had the requisite algebra and calculus skills to solve the nonroutine problems, but were unable to bring them to mind or to use them. This was a concern first raised by the study done with C calculus students (Selden et al., 1989). Adding the numbers in Table 4, one sees that for most nonroutine problem attempts (103 out of a possible 140), the differential equations students had an adequate knowledge base (i.e., full or substantial factual knowledge), yet just 32 of their attempts were successful (i.e., produced completely or substantially correct solutions to nonroutine problems). The inability of many otherwise successful students to access and effectively use their factual knowledge of calculus in nonroutine problem solving is perhaps the most striking feature of our data.

Editorial comment on our second study (Selden et al., 1994) of A/B calculus students raised the question of whether the parameterizations in some of our nonroutine problems might have caused them to be viewed as questions about families of functions, thereby rendering them inordinately difficult. Indeed, three of the five problems *could* be viewed in this way (Problems 1, 3, and 5). However, rather than being interpreted as indicating families of functions, the letters a and b appeared to have been interpreted by these students as fixed unknowns whose values they were expected to find. Much of their written work supports this idea. The a and b seemed to play much the same "arbitrary constant" role for these students as m and b do in the slope-intercept form of a line, $y = mx + b$. In fact, it was on these three problems (especially Problem 5) that the students did the best. Problems 1, 3, and 5 accounted for 85% (17 of 20) of the completely correct solutions.

We conclude that these problems were no more difficult for our students than Problems 2 and 4. If anything, Problems 2 and 4 proved to be the most difficult ones, and it was on these problems that students used algebraic and numerical trial-and-error methods. While it was possible to solve Problem 2 without calculus and three students did so completely correctly, Problem 4 was particularly intractable without calculus. Some students may not initially have accessed their calculus knowledge because this problem brought to mind algebraic techniques for solving an equation, namely $4x^3 - x^4 = 30$. Developing a calculus-based solution would have entailed considering $4x^3 - x^4$, alternatively $4x^3 - x^4 - 30$, as a function to be maximized, something they did not do.

In discussing Problem 4 we have mentioned bringing knowledge to mind, or accessing it, and the next section will depend on this terminology. The knowledge to which we refer is part of what has been called a person's knowledge base (Schoenfeld, 1992; Selden and Selden, 1995). This, in turn, is contained in what cognitive psychologists would call long-term memory (Baddeley, 1995). One is not aware of the contents of long-term memory, but knowledge there can be "activated," i.e., brought into short-term memory, and thereby brought into awareness. This appears to be close to what we mean by knowledge coming to mind. We do not mean just that the knowledge can be acted upon or used, but that it can be used in a special way – that it is conscious. There are a number of kinds of consciousness. Some arise from the external world through the senses after considerable lower-level processing and construction of meaning. Others include inner-speech and vision, or more subtle phenomena such as a feeling of understanding. Differing aspects of consciousness have been discussed by James (1910); for an example of more recent work see Mangan (1993).

5.2.1. *An additional kind of knowledge.* It may be that students who failed to solve our nonroutine problems despite demonstrating full factual knowledge did not effectively employ some of Schoenfeld's hallmark categories of problem-solving, i.e., resources (including knowledge), control, heuristics, and beliefs (Schoenfeld, 1985). However, our problems are not *very* nonroutine and their solutions are relatively straightforward. The unfruitfulness of false starts is not especially hidden, e.g., attempting to factor $4x^3 - x^4 - 30$ in Problem 4; also there is little need to invent multiple sub-problems. Solving such problems seems to call especially on resources, in particular on a kind of knowledge that "triggers" the use of factual knowledge appropriate to the specific problem situation. To put this another way, we suspect that what is required to solve our moderately nonroutine problems is, mainly, some sense of the domain and what is important in it, above and beyond basic mastery of techniques, definitions, and their entailments. It is the way this sense of domain, regarded as a kind of knowledge, can bring to mind factual knowledge that we hope to explain.

The nature of this additional kind of knowledge cannot be fully established from an analysis of data such as ours, which does not emphasize the process aspect of problem solving. However, we will now frame a discussion of it in terms that might be useful in later, more process-oriented, research. A person who has reflected on a number of problems is likely to have seen (perhaps tacitly) similarities between them. He or she might be regarded as recognizing (not necessarily explicitly or consciously) several overlapping problem situations arising from problems with similar features. For example, experienced students would probably recognize a problem as one involving factoring, several linear equations, or integration by parts.

A problem situation seems to be much like a concept. While such a situation may lack a name, for a given individual it is likely to be associated with an image. This image is a mental structure possibly including strategies, examples, non-examples, theorems, judgments of difficulty, and the like, linked to the problem situation. Following Tall and Vinner's (1981) idea of concept image, we call this kind of mental structure a *problem situation image* and regard it as a part of one's knowledge base. When a problem situation is recognized, most of the features in its image do not immediately come to mind, i.e., into consciousness. Rather they seem to be partly activated (Baddeley, 1995), that is, they are very easily brought to mind as needed. Some such images may, and others may not, contain what we will call *tentative solution starts*: tentative general ideas for beginning the process of finding a solution. A tentative solution start might be seen as a strategy for solving the problem at hand. For example, a problem situation image involving the solution of an equation might include "try getting a zero on one side and then factoring the other." It might also include "try writing the equation as $f(x) = 0$ and looking for where the graph of $f(x)$ crosses the x-axis," or even "perhaps the max of f is negative so $f(x) = 0$ has no solution." We suggest that the problem-solving process includes the recognition of a problem as belonging to one or more problem situations and partly activates their associated images. This, in turn, easily brings to mind a tentative solution start or strategy (specific to the problem) contained in one of those problem situation images. This may thereupon mentally prime the recall of additional resources from one's knowledge base. Thus a tentative solution start may link recognition of a problem situation with recall of appropriate resources, i.e., what we have called accessing factual knowledge.

Although the kinds of calculus problem situations perceived by students do not seem to have been well examined, a number of studies of problem-solving performance support the idea that students recognize problem situations in various ways (Hinsley, Hayes, and Simon, 1977; Schoenfeld, 1985, Chapter 8). In research on physics problem-solving, features focused on by solvers have been observed to correspond to the degree of expertise. Novices tend to favor surface characteristics (e.g., pulleys), whereas experts tend to focus on underlying principles of physics, like conservation of energy (Chi, Feltovich, and Glaser, 1981).

In this study a number of students tried factoring on Problem 4. When this algebraic approach did not work, had those students' images of such problems included "try looking at whether the graph crosses the x-axis," they might have recalled their knowledge of graphs and calculus to discover that the maximum was too small for the equation to be solvable. In viewing our data from this perspective, we are suggesting that problem situations, their images, and the associated tentative solution starts all vary from student to student and that the process of mentally linking recognition (of a problem situation) to recall (of requisite resources) through a problem situation image might occur several times in solving a single problem. We are not suggesting this is the only way accessing one's knowledge base might occur, just that it could play an important role in solving the kind of moderately nonroutine problems discussed in this paper.

Recognizing a problem as associated with a particular problem situation image and bringing to mind a tentative solution start from that image results in what Mason and Spence have called "knowing-to act in the moment" (1999). Here the "act" is using the tentative solution start and *knowing to* should be contrasted with *knowing how* (procedural knowledge) and *knowing that* (conceptual knowledge). For example in Problem 4, reading the problem invokes a problem situation image whose richness depends upon the student's previous experiences. A possible tentative solution start that might come to mind from the student's image would be *knowing to* subtract 30 from both sides. However once the student establishes that attempting to solve the equation by algebraic methods is fruitless, the problem situation is changed. If the student's problem situation image is rich enough to contain a tentative solution start based on examining where the graph of $y = 4x^3 - x^4 - 30$ crosses the x-axis, knowing to do so would probably come to mind. We mean by this that the student could unhesitatingly respond if questioned about her/his intentions at that moment, and might report a conscious feeling of intending to examine the graph, or even report having articulated that intention in inner-speech. Alternatively, the student might simply begin to draw the graph. However, if the student's problem situation image lacked this or a similar tentative solution start, he or she might bring nothing to mind and would be reduced to searching her/his knowledge base for a serendipitous link to the problem. Many students may have neither time nor self-confidence enough for such a search. They might not solve the problem even if they know the relationship between roots of an equation and where the graph of the corresponding function crosses the x-axis because such information was not linked to the problem in a way that brings using it to mind.

Except while actually taking classroom tests, students in this study could have consulted worked examples from textbooks or lecture notes during their previous problem-solving attempts. Those who habitually consulted such worked examples before attempting their own solutions may have had little occasion or reason to reflect on multiple tentative solution starts. Such students might well have impoverished problem situation images with very few tentative solution starts, thereby reducing the usefulness of these images in priming the recall of factual knowledge. This could happen even if they realized a new idea was needed, that is, even if they exercised (metacognitive) control (Schoenfeld, 1985, 1992). In summary, we are suggesting that some of the students who were unable to solve our nonroutine problems, although demonstrating full factual knowledge, may have lacked a particular kind of knowledge, namely tentative solution starts. This may have been due to a combination of the text, the way homework exercises were assigned and discussed, and our students' study habits. Much of our data concerning these students, who had adequate factual knowledge but did not access it to solve our nonroutine problems, is consistent with the above analysis.

5.3. Does it matter whether students are able to solve nonroutine problems? Perhaps surprisingly, the answer seems to be both yes and no. No, because the students in this study were among the most successful at the university by a variety of traditional indicators, both at the time of the study and subsequently, yet half of them could not solve a single nonroutine problem. They had overall GPAs of just above 3.0 at the time of the study and almost double the graduation rate of the university as a whole. At least seven of them subsequently earned a master's degree and one a Ph.D. Furthermore, the idea that traditional academic success may not require very much nonroutine problem-solving ability, including metacognitive control, is supported by DeFranco's problem-solving study comparing mathematicians of exceptional creativity (e.g., Fields medallists) with very successful published Ph.D. mathematicians. He found that while both were content experts, only the former were problem-solving experts (DeFranco, 1996). Thus, it seems possible to be academically successful in mathematics and related subjects without consistently being able to solve nonroutine problems, especially the more difficult ones in which Schoenfeld's (1985, 1992) problem-solving characteristics (resources, heuristics, control, and belief systems) play a large role.

On the other hand, yes, it does matter. Most mathematicians in our experience seem to regard this kind of problem solving as a test of deep understanding and the ability to use knowledge flexibly. In addition, most applied problems that students will encounter later will probably be at least somewhat different from the exercises found in calculus and other mathematics textbooks. It seems likely that much original or creative work in mathematics would require novel problem solving at least at the modest level of the nonroutine problems in this study. Thus, when other academic departments decry students' inability to apply their mathematics, the difficulty may lie partly with the ability to solve nonroutine problems generally, rather than with particular kinds of applications.

In addition mathematicians often appear to view students' mathematical ability through the lens of problem solving, meant broadly to include a wide range from simple exercises to nonroutine problems and the construction of proofs. As teachers they typically design problems whose solution requires deep understanding rather than gauging such understanding directly, for example, through evaluating an essay on the nature of continuity and its relationship to differentiability. Thus, in order to more precisely discuss mathematicians' views of which student abilities matter, it would be useful to have a way of analyzing kinds of problems. In what follows, we discuss only degrees of nonroutineness and not the many other facets of problem solving.

5.3.1. *A nonroutineness scale.* It may be helpful to consider problems (tasks together with solvers) arising in a mathematics course as arranged along a continuum according to routineness relative to the course. We regard the degree of routineness of a problem as dependent on what the solver knows so a task that is routine for one student may be nonroutine for another. Nevertheless, the routineness of a problem, for most students, might be estimated by what has come up in the course. At one end of the continuum, we might have *very routine* problems which mimic sample problems found in the text or lectures, except for minor changes in wording, notation, coefficients, constants, or functions, that students view as incidental to the way the problems are solved. Such problems are often referred to as exercises and might not be regarded as problems at all in the problem solving literature (Schoenfeld, 1992). Moving toward the middle of the continuum, we might place *moderately routine* problems which, although not viewed as exactly like sample problems, can be solved by well-practiced methods, e.g., ordinary related rates or change of variable integration problems in a calculus course. Sandra Marshall (1995) has studied how schema can be developed to reliably guide the solution of such problems at and below the precalculus level. Moving further along the continuum, one might have *moderately nonroutine* problems, which are not very similar to problems that students have seen before and require known facts or skills to be combined in a novel way, but are "straightforward" in not requiring, for example, the consideration of multiple sub-problems. The nonroutine test in this study consisted of problems of this kind. Finally, at the opposite end of the continuum from routine problems, one might place *very nonroutine* problems which may involve noticing unusual patterns, considering several sub-problems or constructions, and using Schoenfeld's (1985) characteristics of effective problem solving. For such problems a large supply of tentative solution starts, built up from experience, might not be adequate to bring to mind the knowledge needed for a solution, while for moderately novel problems such as ours it probably would.

Here is an example of what, for most undergraduates, would be a very nonroutine problem[3]. *Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.* This appears to be a calculus problem, but it only requires clever use of geometry. An elegant solution[4] involves extending the construction "outward" by reflecting across both the lines $y = x$ and the x-axis and noticing that the perimeter of the triangle equals the distance along the path from (b, a) to $(a, -b)$. Thus, minimizing the perimeter amounts to making that path straight. Probably only exceptionally experienced geometry problem solvers would have previously constructed images of problem situations containing a tentative solution start that would easily bring to mind such an unusual construction and "path straightening" technique.

Although in this paper we only discuss the degree of routineness of problems, there are other important aspects to problem solving. Some problems require an understanding of the underlying conceptual notions or the application of the core ideas behind the content. In addition, some problems call for the use of heuristics or problem-solving strategies. Furthermore, some problems may require a kind of

[3]Taken from the 59th Annual William Lowell Putnam Mathematical Competition (1998; see also Putnam Exam, 1991;Reznick, 1994).

[4]Posted by Iliya Bluskov to the sci.math newsgroup.

cleverness that goes beyond understanding content and beyond problem-solving heuristics. In particular, we observe that routineness differs from difficulty. The most routine of exercises can be quite difficult, e.g., requiring the fast execution of a long procedure, overloading working memory and leading to what students often call "dumb mistakes."

Most university mathematics teachers would probably like students who pass their courses to be able to work a wide selection of routine, or even moderately routine, problems. In addition, we believe that many such teachers would expect their better students to be able to work moderately nonroutine problems such as ours. Many seem to think of this as functionally equivalent to having a good conceptual grasp and understanding of a course. In other words, the ability to work moderately nonroutine problems based on the material in a course is often part of the implicit curriculum. Thus, yes, it matters if most students cannot work moderately nonroutine problems because for the mathematical community the essence of the material in the course is not being successfully mastered, even by good students.

6. Implications for Teaching

The results reported here suggest that, at least in a traditionally taught calculus/differential equations sequence, many good students may not reach the level of understanding and moderately nonroutine problem-solving ability that their teachers expect. In order that good students reach this level, nonroutine problem solving may need to become an explicit part of the curriculum, that is, to be in some way explicitly taught. Furthermore, our explanation of the data – that students' problem situation images tend to lack a variety of tentative solution starts – suggests that the ability to solve moderately nonroutine problems may depend partly on a rich knowledge of problem situations as well as on more general problem-solving characteristics (Schoenfeld, 1985, 1992).

We limit our comments on teaching mainly to how to improve student ability to solve *moderately* nonroutine problems, e.g., problems similar to those on our nonroutine test. For suggestions on developing student ability to solve *very* nonroutine problems in which all of Schoenfeld's problem-solving characteristics are likely to play major roles, see Arcavi et al. (1998) and Schoenfeld (1985). We will focus on improving a part of what Schoenfeld calls resources, namely the richness of problem situation images. If our analysis in Section 5.2.1, or something close to it, is correct then encouraging students to build rich problem situation images should help bring to mind appropriate factual knowledge when needed. Of course, to bring to mind appropriate factual knowledge when needed, the student must *have* such knowledge. However, it was not factual knowledge that most of our differential equations students lacked, rather it was access to and knowing-to use it. That is what we address now.

For the purposes of this discussion we will assume students know a number of algebraic techniques for solving equations, quite a bit about graphs, the intermediate value theorem, how to find maxima and minima, which functions are continuous or discontinuous, that the real roots of $f(x) = 0$ are the points where the graph of $y = f(x)$ meets the x-axis, etc. The material on our routine test fits into this category. In addition, we assume they know things like solving $f(x) = g(x)$ amounts to solving $f(x) - g(x) = 0$, that the intermediate value theorem can be used to show there is a root to $y = f(x)$, and looking at maxima or minima can show $f(x)$ is entirely positive or negative and thus does not meet the x-axis – material covered in most traditionally taught calculus courses.

To build rich problem situation images that contain a number of tentative solution starts, the students would have to see a problem situation as an object (although probably tacitly and without name) which is associated with other objects. We see student construction of problem situations as objects as requiring a student to engage in multiple activities and experiences. We view this as being analogous to the construction of new concepts, such as the function concept, as described by Breidenbach et al. (1992) or Sfard (1991). Merely pointing out how a solution to a particular problem is started might not be any more effective than merely pointing out the features of a new concept, such as that of function.

One way of giving students multiple experiences that might lead to the construction of rich problem situation images would be to scatter throughout a course a considerable number of problems for students to solve without first seeing very similar worked examples. We mean to suggest a collection of problems that cover the course well and that most of the students really can eventually solve, albeit with some difficulty. The idea is that the students would struggle with these problems and reflect on their solutions more than they would with traditional exercises mimicking worked examples. However, we note that students often expect to be told precisely how to work problems. Thus, a change in the prevalent classroom culture that prefers tedium over struggle and reflection might be required. Such change might be difficult for some instructors, but there is some evidence it can be effected by reiterating in class that struggle and reflection are expected (Davis and Hauk, in preparation).

Alternatively, some problems might be solved in two phases. Problems could be introduced with the understanding that the first phase ends with students reporting, perhaps in writing, on their tentative solution starts, before going on to complete the solution in the second phase. As a practical matter, this kind of activity would probably be taken more seriously if it were in some way reflected in homework, classwork, and tests. For tests, adequate time for reflection would need to be allowed. Indeed, one might not ask for full solutions to such a problems, but only clear descriptions of how to begin.

Another approach to providing the multiple experiences that could lead to rich problem situation images might be to ask students to justify the steps in solved problems, whether from the text, another student, or the instructor. Students might be asked to discuss alternative solution methods and the degree of promise of each. Such an approach might be incorporated into homework and tests and seems particularly appropriate for class presentations by students or group work. As an example, Santos-Trigo (1998) describes an aspect of Schoenfeld's problem-solving course where "homework assignments include problems in which students have the opportunity to discuss differences and qualities of each approach," i.e., to examine tentative solution starts and, "students search for multiple solutions, but they write their ideas clearly and support their methods with mathematical arguments."

A teacher might also use something akin to Socratic dialog with one or several students, or even with a whole class. ¿From the point of view of building rich problem situation images, there are several benefits of such dialog: students can be brought to focus on various tentative solution starts, they can sometimes solve a greater variety of problems than otherwise, and they might eventually adopt the kinds of reflective questions the instructor asks in something like an "inner Socratic dialog." The questions in such an inner Socratic dialog might prompt bringing to mind appropriate factual knowledge. However, such reflective questions themselves *must* come to mind or at least be acted upon. The way inner Socratic dialog might be engendered in students, what form it might take, and to what degree it would be useful are interesting questions for future research. What we are talking about is distinct from the kind of habituation to questions such as "What are you doing?" discussed in Arcavi et al. (1998) and Schoenfeld (1985).

Having students work in groups might combine well with several of the above suggestions. There are a number of benefits to group work but two seem particularly pertinent. Students working in a group have additional sources of factual knowledge beyond those available when working alone. Thus, they may solve more and harder problems (Kieren, Calvert, Reid, and Simmit, 1995; Trognon, 1993). In addition, the discussion in a group may encourage individual students to reflect on their solutions or the groups' solutions. Indeed, discussion itself may be somewhat like reflection (Sfard, Nesher, Streefland, Cobb, and Mason, 1998). Thus, working in groups may help students enrich their problem situation images.

A college instructor's typical response might be, "But I don't have time for this!" We agree that implementing any of the above suggestions may call for considerable time, which is in short supply in college mathematics courses. The time might be obtained by assigning and discussing fewer problems: a reduced collection of problems which addresses a broader range of the nonroutineness scale. Examination of a variety of problems in the ways we suggest, by teacher and students alike, might improve students' ability to solve moderately nonroutine problems without reducing their command of routine exercises.

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