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ILLUSTRATING A THEORY OF PEDAGOGICAL CONTENT KNOWLEDGE FOR SECONDARY AND POST-SECONDARY MATHEMATICS INSTRUCTION

Shandy Hauk	Allison Toney	Billy Jackson	Reshmi Nair	Jenq-Jong Tsay
WestEd	U. North Carolina	U. West	U. Northern	U. Texas –
shauk@wested.org	Wilmington	Georgia	Colorado	Pan American

The accepted framing of mathematics pedagogical content knowledge (PCK) as mathematical knowledge for teaching has centered on the question: What mathematical reasoning, insight, understanding, and skills are required for a person to teach elementary mathematics? Many have worked to address this question in K-8 teaching. Yet, there remains a call for examples and theory in the context of teachers with greater mathematical preparation and older students with varied and complex experiences in learning mathematics. In this theory development report we offer background and examples for an extended theory of PCK – as the interplay among conceptually-rich mathematical understandings, experience in and of teaching, and multiple culturally-mediated classroom interactions.

Keywords: Pedagogical content knowledge, Discourse, Intercultural awareness

Since Shulman's (1986) seminal work, a rich collection of theories and measures of mathematics pedagogical content knowledge (PCK) continues to grow (e.g., Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008; Silverman & Thompson, 2008). However, the work to date on early grades (K-8) teacher development includes little in the way of the classroom sociology and advanced mathematical understandings such as are found in high school and college. There is a need for examples and theory in the context of teachers with greater mathematical preparation and older students with varied and complex experiences in learning mathematics (Speer & King, 2009).

The framing of knowledge for teaching in the K-8 arena has centered on the question: What mathematical reasoning, insight, understanding, and skills are required for a person to teach elementary mathematics? Many have worked to develop measures to address this question, most notably Ball and colleagues (Hill, Ball, & Schilling, 2008). In their work they have defined three types of subject matter knowledge (SMK) and three types of PCK as the domains of "mathematical knowledge for teaching" (p. 377; see Figure 1). Even with their carefully developed model, challenges exist in identifying and measuring PCK (pp. 396-398).

Speer and King (2009) have offered insight into the different demands of a theory of SMK for secondary and post-secondary instruction. We extend the exploration of differing demands and focus on the PCK half of the picture (see Figure 1, at right): knowledge of curriculum, of content and students (KCS), and of content and teaching (KCT).

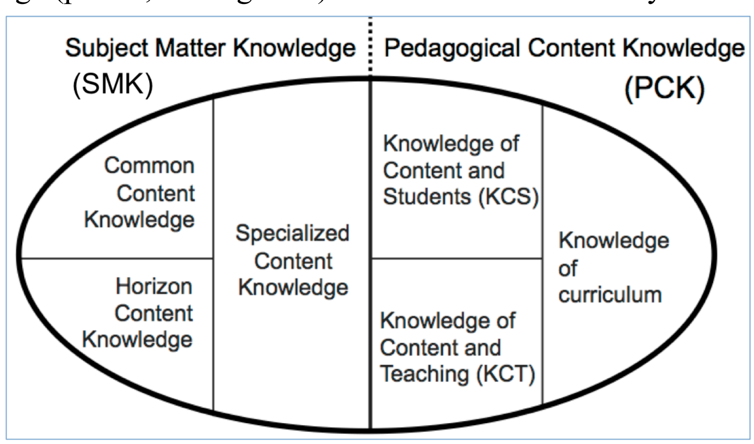


Figure 1. Dimensions of knowledge for teaching from Hill, Ball, & Schilling (2008).

Hill, Ball, & Schilling (2008) acknowledged the problematic nature of identifying types of "knowledge" and have speculated on the need for alternate conceptualizations (e.g., perhaps as "reasoning about") or additional constructs, to capture the multi-dimensional nature of PCK. Other researchers have offered a supplement to the K-8 view, emergent from radical constructivist perspectives (i.e., Piagetian). It is the idea that for some, PCK is "predicated on coherent and generative understandings of the big mathematical ideas that make up the curriculum." (Silverman & Thompson, 2008, p. 502). In this framing, PCK grows when a teacher gets better at the transformation of personal and intimate forms of mathematical knowing. Our purpose in building theory is to describe and illustrate an unpacking of these ideas – attending to people's ways of understanding, thinking, and reasoning about and through mathematics in order to teach, while also attending to the reality of culturally heterogeneous classroom contexts.

Here we report on our efforts to develop an expanded theory and model of PCK that considers a key aspect of Shulman's (1986) original framing that is absent in existing models. Based on work discussed below, it is called *knowledge of discourse*. This brings to PCK the mathematical "syntax" that was part of Shulman's description:

> The syntactic structure of a discipline is the set of ways in which truth or falsehood, validity or invalidity, are established... Teachers must not only be capable of defining for students the accepted truths in a domain. They must also be able to explain why a particular proposition is deemed warranted, why it is worth knowing, and how it relates to other propositions, both within the discipline and without, both in theory and in practice… This will be important in subsequent pedagogical judgments regarding relative curricular emphasis. (Shulman, 1986, p. 9)

Ultimately, we seek to develop theory and measurement tools/guidelines that allow exploration of questions such as: *What is the interplay among mathematical understandings, teaching, and culturally mediated communication in defining and growing PCK?*

Our proposed framework relies on three existing theories related to human interaction in mathematics teaching and learning: for discourse, for intercultural awareness, and for PCK. We start with brief definitions associated with "discourse," make a foray into some key ideas in intercultural orientation, and then describe our model with additional PCK constructs. We conclude with two classroom vignettes and brief analyses of them to illustrate the theorized PCK constructs. These illustrations are *not* definitions. They are offered as anchors for discussion.

Background on d/Discourse

In his review of over one hundred research publications in mathematics education that reported on "discourse," Ryve (2011) concluded that conceptualizations of discourse are varied in detail and diverse in scope. He noted that the field would benefit from explicit definitions for "discourse" each time it is used in reporting research or theory. What Ryve found in common across the reviewed articles was that the conceptions of "discourse" could be understood through the work of Gee (1996), who distinguished between "big D" Discourse and "little d" discourse.

A classroom culture is a set of values, beliefs, behaviors, and norms shared by the teacher and students that can be reshaped by the people in the room (Hammer, 2009). Though not everyone in the classroom may describe the culture in the same way, there would be a general center of agreement about a set of classroom norms, values, beliefs, and behaviors. Whereas Gee's (1996) "little d" discourse is about language-in-use (this may include connected stretches of utterances and other agreed-upon ways of communicating mathematics such as symbolic statements or graphs), Discourse ("big D") includes little d discourse *and* other types of communication that happen in the classroom (e.g., gestures, tone, pitch, volume, and preferred ways of presenting information). The forms of communication in discourse are usually explicit and observable, while the culturally embedded nature of communication in Discourse is largely implicit. Gee's Discourse also includes Shulman's attention to syntax:

> A Discourse is a socially accepted association among ways of using language, other symbolic expressions, and 'artifacts', of thinking, feeling, believing, valuing, and acting that can be used to identify oneself as a member of a socially meaningful group or 'social network', or to signal (that one is playing) a socially meaningful 'role' (p. 131).

That is, as part of PCK, there is knowledge for working effectively with the multiplicity of Discourses students, teacher, curriculum, and school bring into the classroom. Each Discourse includes a cultural context. Discourses may differ from person to person or group to group. The ways that teachers and learners are aware of and respond to multiple cultures is a consequence of their orientation towards cultural difference, their *intercultural orientation*. We come back to intercultural orientation after unpacking what we mean by Discourse a bit more.

The "big D" Discourse of academic mathematics values particular kinds of "little d" discourse. Valued inscriptions are logico-deductive (e.g., proof) and figural (e.g., representations such as graphs of functions or diagrams of relationships or mappings). Especially valued in advanced mathematical discourse are explanation, justification, and validation (Arcavi, Kessel, Meira, & Smith, 1998; DeFranco, 1996; Weber, 2004). As in other fields, instructors ask questions to evaluate what students know and to elicit what students think. For instance, a model of classroom interaction common in the U.S. is the dialogic pattern of *initiation –response – follow-up* or *I•R•F* structure (Mehan, 1979; Wells, 1993). In college classrooms, this is most often initiated by teachers, but not exclusively so, and the (implicit) rules for how initiating, responding, and following-up will happen are worked out by the people in the room (Nickerson & Bowers, 2008). These rules make up one aspect of what Yackel and colleagues have called "socio-mathematical norms" (Yackel, Rasmussen, & King, 2000).

In his ethnographic work, Mehan identified four types of teacher questions (see Table 1). Research suggests that U.S. mathematics instructional practice lives largely to the left of Table 1 (Stigler & Hiebert, 2004; Wood, 1994). The unfortunate aspect here is not the fact that evaluative questions are common but that the eliciting questions, in the right column, are not. These more complex spurs for discourse can lead to iterative patterns that cycle through and revisit the frame of reference "in ways that situate it in a larger context of mathematical concepts" and foster "mathematical meaning- making" (Truxaw & DeFranco, 2008, p. 514). The use of process and metaprocess questions, for example as *follow-up (F)*, readily expands discourse into the "reflective toss" realm of comparing and contrasting different ways of thinking (with justification but without judgment), monitoring of a discussion itself, as well as attending to the evolution of the thinking of others and self (van Zee & Minstrell, 1997).

Table 1. Initiate-Respond-Follow-up (*I•R•F*) question types and anticipated response types.

Evaluate what students know	***Elicit what students think***
Choices – response constrained to agreeing or not with a statement (e.g., Did you get 21?)	***Processes*** – response is an interpretation or opinion (e.g., Why does 21 make sense here?)
Products – response is a fact (e.g., What did you get?)	***Metaprocesses*** – response involves reflection on connecting question, context, and response (e.g., What does the 21 represent? How do you know?)

Another important aspect of Discourse is in interaction for teaching. Piaget identified assimilation and accommodation as two interactive processes to explain an individual's adaptation to achieve cognitive equilibration and learn (Driscoll, 1994). Humans are pattern-seekers looking for patterns to recognize for assimilation. If assimilation fails, people may create their own interpretation of ideas, based on available perceptions, for accommodation. From this perspective, teaching is the act of providing productive cognitive conflict so that learners may accommodate their existing schemes, iteratively, in ways that incorporate rigorous mathematical schemes. That is, concept images are challenged repeatedly by cognitive disequilibration to foster the development of the associated concept definition (Tall & Vinner, 1981). This is in contrast to pseudo-assimilation (e.g., about function; Zandieh, 2000).

As an example, consider the types of pseudo-assimilation discussed by Bair and Mooney (2013). They offered examples of problematic instruction on the distributive property such as "FOIL" and "bam-baming" two negative signs to a positive in an expression like $4 - (6 - 3x)$. Although aiming to reduce what learners may find cognitively overwhelming, these may lead students to unproductive generalizations and counter-productive decisions about mathematical meanings. Similarly, Temple and Doerr (2012) note the importance of developing fluency in the mathematical register – thought and speech inform each other and using technical vocabulary can support mathematical meaning-making. Discourse is central in our effort to bring to PCK theory an explicit attention to the use of language and the dense set of values about mathematical appropriateness, clarity, and precision that are integral to thinking, learning, and communicating mathematics. In what follows, we use d/Discourse in Gee's (1996) culturally informed way.

Intercultural Orientation

The construct of "big D" Discourse as part of mathematics PCK pivots on the idea of intercultural orientation. Our referent framework is the *Developmental Model of Intercultural Sensitivity* (Bennett & Bennett, 2004). The developmental continuum of orientations towards awareness of cultural difference, of "other," runs from a monocultural or ethnocentric "denial" of difference based in the assumption "Everybody is like me" to an intercultural and ethnorelative "adaptation" to difference. The first move, from denial to the "polarization" orientation, comes with the recognition of difference, of light and dark in viewing a situation (e.g., Figure 2a).

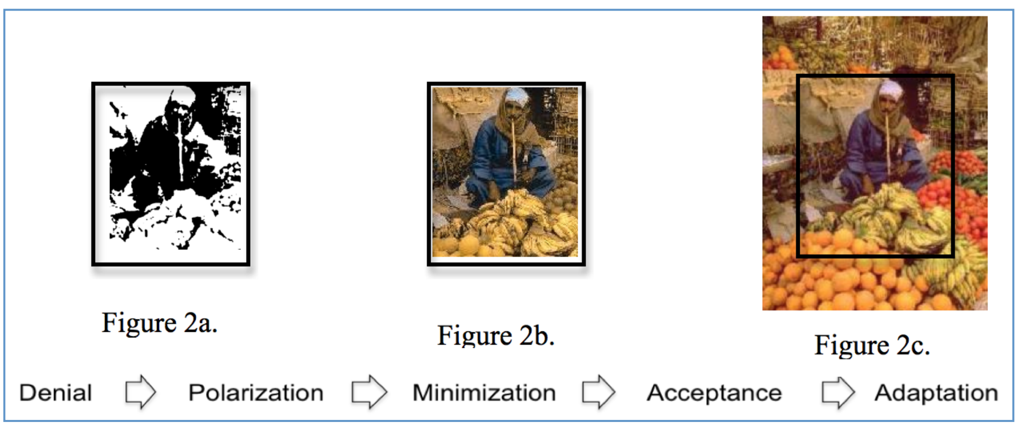


Figure 2. Intercultural orientations and developmental continuum.

The polarization orientation is driven by the assumption "Everybody should be like me/my group" and is an orientation that views difference in terms of a stark "us" and "them." Evaluative prompts about student thinking (left side of Table 1) are more likely for this orientation. Moving along the continuum towards ethno-relative perspectives leads to a minimizing of difference, focusing on similarities, commonality, and presumed universals (e.g., biological similarities – we all have human brains so we all learn math essentially the same way; and values – we all know the difference between right and wrong and naturally will seek right). This is the "minimization" orientation. A person with this orientation will be blind to recognition and appreciation of subtleties in difference (e.g., Figure 2b, a representation of, literally, the view of a colorblind person). The minimization orientation tends to take the form of ignoring fine detail in how people might have differing ways of thinking. For example, efforts at eliciting d/Discourse (right side of Table 1) may take the form of *listening for* particular ways of thinking. Transition from a minimization orientation to the "acceptance" orientation involves attention to nuance and a growing awareness of self and others as having culture and belonging to cultures (plural) that may differ in both obvious and subtle ways. While aware of difference and the importance of relative context, how to respond and what to respond *in the moment of interaction* is still elusive. From this orientation, classroom d/Discourse may include process and metaprocess prompts, but sustained cycles of such interactions can be challenging to maintain in the immediacy of dynamic classroom conversation. The transition to "adaptation" involves developing frameworks for perception, and responsive skills, that attend to a spectrum of detail in an interaction (e.g., the detailed and contextualized view in Figure 2c). Adaptation is an orientation where one is aware of multiple relative perspectives, and may – without violating one's authentic self – adjust communication and behavior in contextually appropriate ways. There is an instrument for measuring general intercultural orientation (see idiinventory.com). The central idea here is that such orientations are learned, developmentally (Bennett, 1993, 2004; DeJaeghere & Cao, 2009).

Extended Model of Pedagogical Content Knowledge

While Hill, Ball, and colleagues took a classical measure theory approach to identifying and assessing teacher knowledge, we continue to investigate a non-linear alternative (i.e., instead of the traditional linear methods such as hierarchical linear modeling). In particular, our current approach focuses on PCK in terms of four areas of professional understanding: Knowledge of Discourse, Curricular Thinking, Anticipatory Thinking, and Implementation Thinking. These four areas connect in many ways with the Knowledge of Curriculum, Knowledge of Content and Students (KCS), and Knowledge of Content and Teaching (KCT) from Figure 1. They differ, however, in that each is a kind of proceptual understanding (Gray & Tall, 1994), with thinking that integrates relational components along with instrumental ingredients (Skemp, 1976). We seek to identify, prompt for, and assess the connected and overlapping relational aspects, especially in how the three types of thinking (curricular content, anticipatory, implementation) interact with knowledge of curriculum, KCS and KCT to be generated by and generative of Knowledge of Discourse.

Hill, et al. (2008) acknowledge the importance of teacher knowledge of standard and non-standard mathematical representations and communication, but knowledge of d/Discourse as we construe it – composed of discourse and Discourse – does not appear explicitly in their model. One way of visualizing our extension, that highlights and focuses on the interplay among the components of the new and existing models, is as the surface of a tetrahedron whose base is the existing model with a new vertex of Knowledge of Discourse (see Figure 3). We have focused on knowledge of discourse and the three "edges" connecting it to the components in Figure 3 (Hauk, Jackson, & Noblet, 2010). These edges are labeled as "ways of thinking" in the sense of Harel (2008). We continue to explore the possibility of the "knowledge of" areas being taken as "(ways of) understanding" (Harel, 2008).

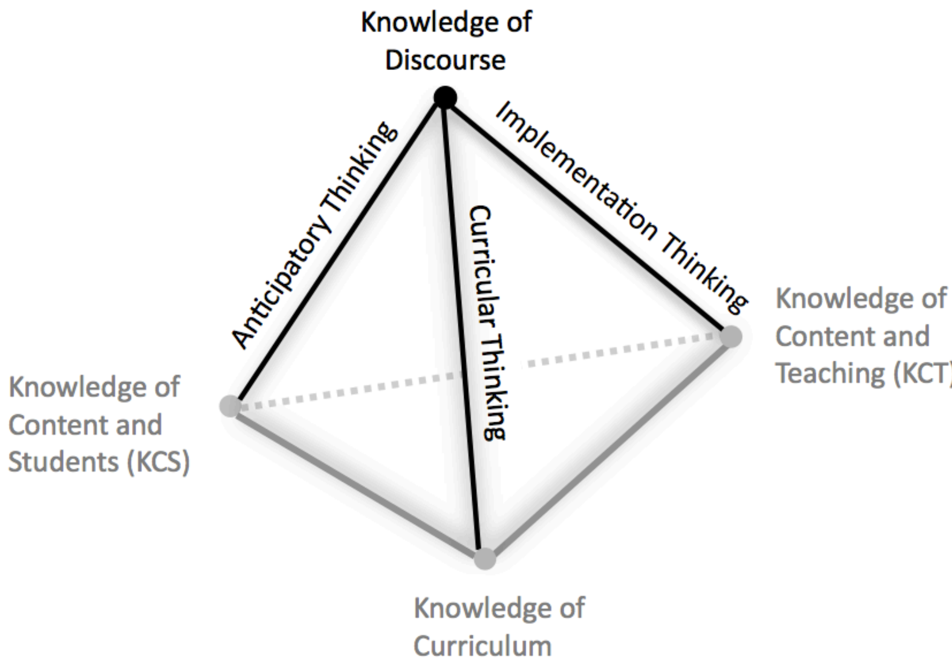


Figure 3. Tetrahedron - vertices, edges, and surfaces - as a way to visualize PCK components and relationships. Corners of the base are PCK dimensions from Figure 1.

Knowledge of Discourse is d/Discourse knowledge about the culturally embedded nature of inquiry and forms of communication in mathematics (both in and out of educational settings).

Curricular Thinking is ways of thinking about mathematical topics, procedures, and concepts as well as the relationships among them, and conventions for reading, writing, and speaking them, in curricula. In its most robust form, this part of PCK contributes to what Ma (1999) called "profound understanding of mathematics" (p. 120). In combination, curricular content and d/Discourse are the home of Simon's (2006) "key developmental understandings."

Anticipatory Thinking is ways of thinking about (strategies, approaches to) how learners may engage with content, processes, and concepts. It includes awareness of and responsiveness to student thinking. Part of anticipatory development involves what Piaget called "decentering" – building skill in shifting from an ego-centric to an ego-relative view for seeing or communicating about an idea or way of thinking from the perspective of another (e.g., eliciting, noticing, and responding to student thinking; Carlson, Moore, Bowling, & Ortiz, 2007). Teachers with complex anticipatory thinking manage the tensions among their own instrumental and relational understandings of mathematics and its learning and those of their students (Skemp, 1976). Such perspective-shifting is deeply connected to d/Discourse through the awareness of "other" as different from "self." We see this as intimately connected to intercultural orientation.

Implementation Thinking. This is ways of thinking about (strategies, approaches to) how to enact teaching intentions in the classroom. Moreover, for us, it includes how to adapt teaching according to content and socio-cultural context and act on decisions informed by d/Discourse as well as curricular content and anticipatory ways of thinking. We do not argue for an intention to enculturate in the sense of Kirshner's (2002) "teaching as enculturation" (i.e., to identify a reference culture and then target instruction for students to acquire particular dispositions). Nor do we propose his alternate framings (habituation, construction) or any other preference for a particular implementation paradigm.

Vignettes and Discussion

Over the last 10 years, the authors have been involved in a variety of ways in research and professional development with post-secondary faculty, in-service secondary mathematics teachers, and their students. In that work, mathematically trained stakeholders regularly ask us for examples and non-examples of PCK in use. The two vignettes included here are about Teacher Pat, a mathematics doctoral student at a large university whose teaching assistantship includes leading a recitation section in undergraduate abstract algebra. The examples are based on real classroom transcripts from various research projects by colleagues and ourselves. Vignette 1 is Teacher Pat in the first year of teaching a group theory recitation.

Vignette 1 – Snapshot of an Abstract Algebra Classroom

Pat stands in front of the whiteboard. Twenty students are seated among 36 small desk-chairs arranged in 6 rows of 6, all facing the front of the room. Problem on the board reads: *Let G be a group of even order. Prove or disprove that there exists a nonidentity element x of G such that* $x^2 = e$. Shuffling of paper, scratching of pencils, but no voices as students work.

Pat: Okay. Let's talk about this problem a bit. We've been talking lately about planning out our proofs – to decide if we believe the statement to be true before proving it. How is it that you thought about this proof?

Lee: I made a set with an even number of elements and then wrote the inverses. One to two, two to three, and so on.

Pat: Okay *(pause)*, what did you do when you got to the end of your set?

Lee: That went back to one.

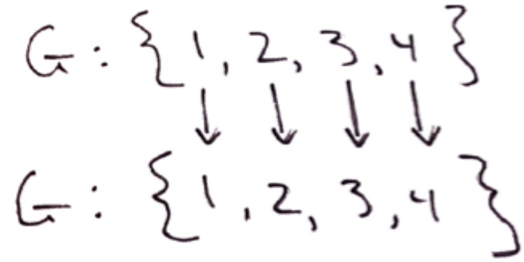
Pat: So, you disproved the statement using a counter example?

Lee (appearing puzzled by the question; looks down at own work; looks back at the problem on the board): Um-

Pat: But this statement is actually a true statement, right? *(To the class)* Right?

(Pat writes "G: {1, 2, 3, 4}" twice on the board, vertically stacked; draws an arrow from 1 on the top line to 1 on the bottom line, and then draws similar arrows for 2, 3, and 4.)

Pat: Here's a set, G, of even order – it has an even number of elements. One goes to one, two to two, three to three, four to four. Each element is its own inverse.



So, we see that the statement can be true.

Lee: Oh! Or can three and four can be each other's inverses and one and two still each be their own inverse?

Pat: Right. As long as the group has an even order, there is always at least one element, in addition to the identity, that is its own inverse. *(Pat looks around the room)* Any questions? Let's write out the proof. *(Turns and erases the board).*

Jackie (quietly, to self) But why couldn't it go around a circle?

Pat: What's that? *(Turns to face the room)*

Jackie: Couldn't one go to two, two to three, three to four, and four back to one?

Pat: No. Then the inverse of the identity wouldn't be the identity. The inverse of the identity always has to be the identity.

Jackie (shrugging): Um, okay.

Pat: Okay. Does anyone have any more questions? Okay. Now that we've planned out our proof, let's write it up formally.

Figure 3. Vignette representing Teacher Pat's instruction in first year of teaching.

Vignette 2 is Pat teaching the same kind of section, after two years that included observing others' classes and participating in seminars about noticing and responding to student thinking.

Vignette 2 – Snapshot of an Abstract Algebra Classroom

Pat stands in front of the whiteboard. Twenty students are seated in small desk-chairs scattered around the room; some are in rows facing the front, while others are clustered facing each other in groups of 2-3. Problem on the board reads: *Let G be a group of even order. Prove or disprove that there exists a nonidentity element x of G such that* $x^2 = e$. Shuffling of paper, scratching of pencils, rumble of voices as students work and talk to each other.

Pat: Let's talk about this problem a bit. From what I could tell walking around the room, some of you found this proof challenging but doable, and some of you found it challenging and struggled to come to a proof. *All* of you found it time consuming *(students nod and a few look at each other and chuckle)*. We've spent some time recently talking about planning out your proofs – to decide if you even believe the statement before tackling it. Let's take a look at somebody's work who feels like they planned out their proof. Could I get a volunteer? Or volunteers, for those of you who worked together?

(Students look around the room at one another; Sam and Lia, whose desks are turned to face each other, make eye contact, Sam raises his eyebrows questioningly, and Lia nods back)

Sam: We'll do it. *(Pat steps aside and gestures to the whiteboard; Sam and Lia come to the board; everyone moves to see the whiteboard)*

Sam (picking up a marker and writing on the board): We started by letting G be this set. *(Sam writes, "G: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}")*

Pat: Okay wait. Can you say why you wrote G as this set of elements? The statement doesn't say anything about a specific set.

Lia: Well, we know G is a group of even order. That means G has an even number of elements. So, as part of our plan we decided to suppose G is this set *(Lia points to the set written on the board).*

Pat: Do you all agree with Lia? *(Students nod or say "yes")*

Sam (writing the same line "G: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}" below the first line): We can write the inverses below, *(Sam draws an arrow from 1 on the top to 2 on the bottom),* where one goes to two. And then we can keep going like this: two to three, three to four, and so on *(Sam adds arrows connecting n on top to n+1 on bottom)…*

So, this means that the statement is true and we think we're ready to start proving.

Pat (to the class): Are there any quest- *(Pat interrupts himself)*- I mean, *what* questions do you have for Sam and Lia?

Jonah: Could you say what the arrows mean?

Lia (pointing to Sam's lists on the board): One goes to two, two goes to three, and so on.

Pat: Yes, but could you please explain what "goes to" *means*?

Sam: Inverses.

Dana: So, one is the inverse of two and two is the inverse of three?

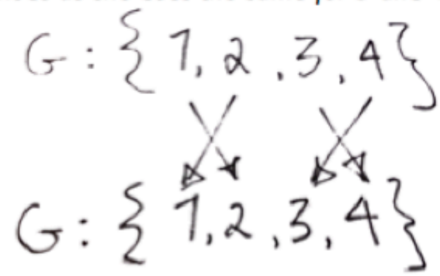
Pat (to class): Do you all agree?

Jonah: No, because then two's inverse would be one.

Pat (to the class) Is that problematic? Think about this for a minute.

(Rumble of voices, shuffling of paper, scratching of pencils; Sam and Lia turn and talk to one another; Lia erases and writes "G: {1, 2, 3, 4}" twice, vertically stacked, and draws arrows

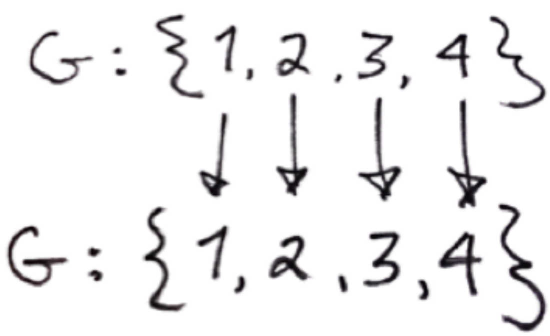
crossing from 1 on top to 2 on the bottom and from 2 in the top line to 1 in the bottom; Sam nods as she does the same for 3 and 4)



Sam (to Pat): We just disproved the statement using a counter example.

Pat (to the class): Let's turn our attention back to Sam and Lia. They say they've found a counter example. *(to Sam and Lia)* Okay, so which element was the identity?

Lia: Well…uh… *(Pause)* The inverse of the identity is the identity!
(She quickly erases the lines and draws a new arrow from 1 on the top line to 1 on the bottom line denoting 1 as the identity)



Lia: Each of the four elements is its own inverse. *(She draws arrows for 2, 3, and 4)*

Nasir: Wait!

Pat: What it is Nasir?

Nasir: (hesitatingly) Can't three and four still be each other's inverses?

Pat (looking around): What do others think?

Dana: Yes! And then two would have to be its own inverse!

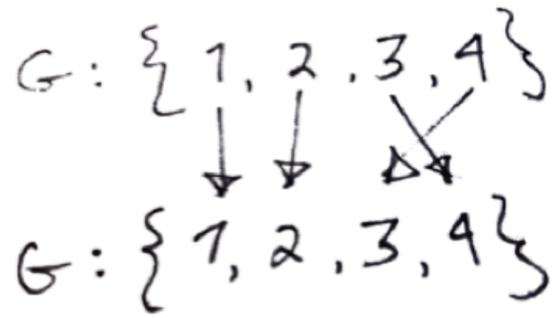
Pat: Slow down. Can you explain Dana?

Dana: Three and four can be each other's inverses. We know one is the identity and it has to be its own inverse. So, two would also have to be its own inverse.

Bo (to Dana): Why would two have to be its own inverse?

Dana (turning to look at Bo): Well, because the order of the group is even.

Lia: So, what you're saying is it could also look like this. *(Lia turns to the board, erases the arrows from 3 and 4 in the top and redraws, crossing them.)*



Lia: Therefore, as long as a group had an even number of elements like in here *(Lia adds a vertical dotted line slicing between 2 and 3 in both sets)*, there would always be *at least* one element in addition to the identity that had to be its own inverse.

Pat: Okay. Thanks, Lia. Thanks, Sam. You two can go ahead and sit down. Did everyone follow that? Everyone take three minutes now to revisit what you have written, and when you're ready we'll get a draft of the formal proof on the board.

Figure 4. Vignette of Teacher Pat's classroom instruction in third year of teaching.

As part of the responsibilities for leading a recitation section, the lead professor with whom Pat worked, Dr. Gold, required graduate students to sit in on the main class meetings. For the interested reader we also offer an online appendix (Toney, Hauk, & Hsu, 2013) – a snapshot of Professor Gold's classroom on the first day of the semester. The idea of the appendix vignette is that Professor Gold works from the first day to establish classroom social and sociomathematical norms that are dialogic. Gold's voice is not the only voice in the room and Gold works to have students feel comfortable with their voices being heard (e.g., orchestrating whole class discussion by asking for raised hands, pausing, requesting think time, and then asking for hands again). Additionally, the professor's comfort with the curriculum is such that Gold effectively anticipates student struggles with group theory concepts. Starting on the first day, instructional materials support student-centered development of concepts. By the time of Vignette 2, Teacher Pat has spent several years observing Gold, someone whose classroom practice is aligned with the right side of Table 1. The reader is encouraged to read through both Pat vignettes before going on to the discussion below. Reading the appendix vignette from Dr. Gold's class may give helpful information on the kind of instruction that Pat observed.

Knowledge of Discourse. In Vignette 1, Teacher Pat foregrounds the correctness of a way of thinking about mapping out a proof and a single path to that proof. That is, the primary discourse (little "d") in the classroom is largely univocal: Pat's utterances to identify a correct proving procedure. Discourse (big D) is also centered with the teacher, as the explanations valued in the classroom are Pat's. In Vignette 2, Pat repeatedly asks students to explain their thinking and has established an atmosphere where students give and ask for explanations. The utterances in the room are more dialogic than in Vignette 1. To participate in discourse (little "d"), responding students have been asked to offer their own thinking to provide a convincing argument. Eliciting questions by Pat are much more in evidence in Vignette 2. An aspect of the classroom Discourse, then, is that engaging in deep explanation is an expectation of all. The request for and use of student-generated figures on the board is part of the mathematical Discourse as much as the valued behavior of students convincing themselves before presenting a "correct" proof. Common to both vignettes is the use of "goes to" as informal language for "maps to." This is an informal phrase widely used and accepted in advanced mathematics discourse. As noted in the section on d/Discourse, use of informal language constrains and supports learning. It can enhance retention, but may also undermine conceptual accommodation. Students' unchecked use of "goes to" as acceptable mathematical language could result in mathematical inaccuracies. In Vignette 1, Student Lee begins with an abbreviated version of "goes to," which Pat reinforces by using the informal phrase a moment later. While Pat may have a proceptual understanding of mappings, the students may still be juggling process and concept as separate, dis-integrated, mental structures. Use of the phrase may funnel all into the process and bypass connected schematization of "mapping elements to their inverses." In Vignette 2, Pat has the students pause and clarify their meaning to some extent, but a firm disequilibration and clear resolution for students is not portrayed in the piece of class we see in the vignette. In Vignette 1, Pat implements *choice* and *product* questions. If these questions dominate a teacher's contributions to discourse, then multiple disconnected *I•R•F* interactions can yield a teacher-regulated kind of interaction that does not include deep participation by students. This can be true even in inquiry-based instruction (Nassaji & Wells, 2000; Wertsch, 1998).

Curricular Thinking. There are subtle and distinct differences between the two vignettes with respect to Pat's content questioning. In Vignette 1, Pat's responses include immediate correct or incorrect feedback. Pat also mentions briefly the idea of a larger goal of planning a proof, while an integration of underlying rationales for such planning is implicit. Unlike Vignette 1, in Vignette 2, Pat's questioning provides cognitive conflict about central concepts (identity, inverse, and to some extent, mapping). To resolve the dissonance, students attend to the properties of identity and inverse, and also notice their interaction (the mapping). A potential connection to the next curricular step lurks in the background as Pat ends the segment by directing students to reflect on what they think and opens the door to connecting it to proof.

Anticipatory Thinking. In Vignette 1, Pat demonstrates anticipatory thinking (and *I•R•F* evaluative approval) of a correct proof path expressed as procedural knowledge. Additionally, Pat does not appear to anticipate the variation in student thinking in the room. Based on Lee's explanation, Pat asserts a misconception for Lee (though Lee seems unaware of it). A moment later, Pat evaluates Jackie's statement rather than taking up the statement as an anticipatory opportunity about student confusion. That is, in Vignette 1, Teacher Pat does not appear to anticipate common student struggles, while also noting a (possible) struggle in a way that is not especially productive. This leads to a question about the nature of anticipatory thinking and its relationship to what actually happens in the classroom (i.e., how might anticipatory thinking be seen as subtly and grossly different from implementation thinking). As we see in Vignette 2, anticipation can be a valuable resource for enhancing students' understanding of mathematics. In Vignette 2, Pat asks guiding questions that involve student thinking. Also, Teacher Pat anticipates that students may possess some knowing of properties of identity and inverse, but may not recognize their interaction in the context of the particular proposition in question. Pat looks to elicit an intellectual need for accommodation by having students display and consider the potential mismatch of information through the representations they draw. In Vignette 2, anticipating and eliciting of student thinking are central and are leveraged by Pat as an implementation strategy: students make sense of their rationales as part of proof planning.

Implementation Thinking. Vignette 1 indicates Pat has a proof map in mind to guide steps of an example and Pat's implementation thinking includes putting Pat's idea of a correct solution path into the air in the room. While Teacher Pat's own subject matter knowledge may operate WLOG (without loss of generality), that strategy may not be familiar to or understood by students. That is, Vignette 1 is pedagogy for proceduralizing proof writing (e.g., "What did you get when you got to the end of your set?"). There is no student-to-student interaction and when Pat overhears Jackie's question, the response is to evaluate and correct (staying to the left of Table 1). In Vignette 2, Pat actively elicits and connects student thinking to procedures *and* concepts. Pat's implementation encourages students to make sense of each other's ideas. As students present their ideas, Pat emphasizes reasoning rather than the product (e.g., "Can you say why you write *G* as this set of elements?"). Pat also uses multiple modes of discourse, including student generated representations and confirming questions in order to support the needs of various students. Teacher Pat asks the students to clarify their terminology and language so others can make sense of it and share their understanding (e.g., "Could you explain what 'goes to' means?"). The connection between what has been put on the board and what "writing a proof" means remains unspecified at the end of Vignette 2. Pat's implementation thinking in Vignette 1 focused on getting the right answer in the air whereas Pat's implementation approach in Vignette 2 seems to incorporate aiming for the next curricular step, attending to student thinking, and building effective discourse through attention to making sense of and reasoning about the mathematics at hand.

Conclusion

Researchers have suggested that some forms of effective teaching may be comparable to improvisational performance (Borko & Livingston, 1989; Bourdieu – see Grenfell & James, 1998; Yinger, 1987). Teaching requires complex management of instructional resources, including the teacher's own subject matter and pedagogical content knowledge. How communication is initiated, normed, and revised in the classroom is shaped by intercultural awareness. We have attempted to capture and include the shaping of classroom mathematics communication in an extended theory of PCK as *Knowledge of Discourse*. It is likely that rich knowledge of discourse would be fundamental to the kind of teaching that can be characterized as effective improvisation. Success is not just about what is said, but also how it is said, as well as the intimacy established among the participants in an improvisational interaction. In the mathematics classroom, teaching extends beyond precise and accurate transmission of facts or uptake by students of information. Rather, it includes taking into account the background and experiences (mathematical and otherwise) of the people in the room, and making decisions informed by that knowledge and instructional context to shape opportunities for learning.

An area of ongoing work for us is the relationship between intercultural orientation and what orientation(s) may be necessary, if not sufficient, for rich d/Discourse development for teaching. In particular, we continue to explore the extant literature on the concept of "decentering" as one potential instantiation of the developmental intercultural continuum that might be seen at work in classrooms. Moreover, the visualization of the extended theory as the vertices, edges, and faces of a tetrahedron may offer a way of articulating how intercultural orientation, as part of d/Discourse, may be seen (tacitly or overtly) in looking at PCK. That is, suppose each of the four faces in Figure 3 represents a multi-dimensional interaction. For example, consider the face at the back of Figure 3; if we label the "edge" between KCT and KCS (perhaps call it *balancing intended and achieved concepts*) then – if we can go this far without breaking the usefulness of the visual model – how might instructional activity near the lower edge of the face be different from instructional activity on the same face, but closer to the Knowledge of Discourse vertex? Perhaps the difference is the nature of decentering. Or, perhaps it is a more complex intercultural constellation of which decentering is part. Conversely, in comparing Vignettes 1 and 2, where might we point or trace a path on the tetrahedron to indicate that Pat built skill in generating and sustaining conceptually focused discourse during instruction?

While the vignettes included here were for a relatively novice college instructor, at the RUME 2013 meeting, presenters and audience members also talked about the situation where a professor is instructor to a room full of in-service secondary mathematics teachers. In our research with a set of mathematics PhD faculty, the distribution of orientations across the developmental continuum pictured in Figure 2 has been centered in minimization with small variance. At the same time, though distributed more widely across the developmental continuum, the in-service teachers in our work (over 100) have intercultural orientations centered at polarization (Hauk, Yestness, & Novak, 2011; faculty and teachers completed measures of intercultural orientation). We have seen many in-service teachers ready to think about and pay attention to how others' approaches to learning might differ from their own. Meanwhile, their professors have tended to minimize difference. So, when teacher-learners spoke in class about their mathematical understandings and how they differed, professors suggested it was most important to see how the approaches were essentially the same. That is, a challenge for the professors was how to notice *nuances* in the differences across teacher-learners' ways of thinking and use that information in their own anticipatory and implementation thinking. Faculty whose instruction of undergraduates looked like Pat in Vignette 2, were more like Pat in Vignette 1 when working with in-service teachers. The diversity of background and content knowledge is much greater in the teacher-learner population than is typical among undergraduate math majors. It may be that the intercultural pressures on Knowledge of Discourse can be so large as to impede flow along the anticipatory and implementation thinking edges of the tetrahedron. A complementary area for research that might illuminate the relationships is looking at the classroom interactions for polarization-centered in-service secondary teachers. A polarization orientation means identifying difference is a ready skill, but identifying and building on commonality is a challenge. We continue to explore what it means to have rich Knowledge of Discourse and how it and orientation towards cultural difference can support teaching that balances and engages with myriad cultures in-the-moment to scaffold effective mathematical communication among all in the room.

Finally, two suggestions for our ongoing work arose out of the lively discussion at the RUME 2013 session. One was the recommendation that development of the theory presented here pursue the distinction between "ways of thinking" and "ways of understanding." Also arising in discussion at the meeting was the suggestion that we consider a further extension of the visual model with the addition of another tetrahedron for SMK, linked to the PCK model through the Knowledge of Discourse vertex. It is still an open question whether this linking could be useful in thinking about, describing, and developing the knowledge used for teaching mathematics.

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